

Life-Cycle Theory for Human Beings ^{*†}

Antoine Bommier

University of Toulouse (GREMAQ) and CNRS

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Abstract

Human beings are sure to die but do not know when they will die. This paper proposes a general formulation of life cycle theory that accounts for these two fundamental aspects of human life. We stress in particular the role of intertemporal correlation aversion which it is a key concept to understand the role of mortality risks in intertemporal choices.

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[†]All correspondence should be sent to Antoine Bommier, GREMAQ, Université des Sciences Sociales, Manufacture des Tabacs - bat F, 21 Allée de Brienne, 31000, Toulouse, France. E-mail: Antoine.Bommier@univ-tlse1.fr

1 Introduction

It is difficult to think of a reasonable theory of human intertemporal behavior that would not account for the finiteness of life and the randomness of the occurrence of death. Looking into the past, it appears that economic theory has followed a prudent path, solving one difficulty at a time. The first models of intertemporal choice to be formalized, as with Samuelson (1937), simply assumed that life is endless. The finiteness of life was first introduced in the life cycle theory of Modigliani and Brumberg (1954), under the assumption that the length of life is known with certainty. Random death was eventually considered in Yaari (1965), which remains the model of reference to account for uncertain lifetime.

This paper, together with Bommier (2005), aims at showing that this historical development has led to focus on very particular assumptions. Indeed, it is argued that 1) the assumption that agents have pure time preferences has little empirical support 2) intertemporal correlation aversion (referred to hereafter as “ICA”) is another aspect of individual preferences that deserves a special attention.

The present paper and Bommier (2005) differ in their approach. Bommier (2005) introduces a particular specification of lifetime utility functions that rules out the existence of time preferences and shows that it has interesting features, including its ability to model all forms of impatience. The standard argument for assuming that agents have pure time preferences simply crumbles. Human impatience is not evidence of the existence of pure time preferences (since impatience may also result from the combination of risk aversion and mortality), and the robustness of the dominant theory is questioned. In the current paper, we suggest a formal reconstruction of life cycle theory, with the basic requirement that the theory has to account for the finiteness of life and lifetime uncertainty. Instead of following the incremental steps observed in the historical development of life cycle theory, we directly address the following question: how to define a general framework for modelling intertemporal choice of agents who know that

they will die, but do not know when they will die? Of course, the wish to remain as general as possible is balanced with the need to end up with a concrete formulation. Some minimal assumptions are thus needed: we shall assume that individual preferences can be modeled within the framework developed by von Neumann and Morgenstern (1944) and are stationary.

Our first contribution consists in providing an explicit and tractable representation of individual preferences satisfying these assumptions (Theorem 1). Several interesting subsets can be delimited within this wide class of preferences. One option consists in following the route initiated by Yaari (1965), which involves assuming that agents are indifferent to intertemporal correlation but may have pure time preferences. The symmetrical route entails ruling out the existence of pure time preferences and introducing ICA, as in Bommier (2005). Intermediate but more complex routes accounting for both time preferences and ICA are also depicted.

The second theoretical contribution of the paper is that it formalizes the link between fundamental properties of individual preferences and what can be inferred from the behavior of human (and therefore mortal) beings. In particular, we elucidate the various factors that drive human impatience. Theorem 2 provides an explicit and intuitive decomposition of the rate of discount that disentangles the effects of pure time preferences from those generated by mortality and ICA. This suggests ways for measuring time preferences and ICA.

The third contribution of the paper lies in numerical simulations based on actual mortality data that show the interest of accounting for ICA. In particular, we compare the predictions of the standard model with pure time preferences and no ICA with those of the model with ICA but no time preferences, when applied to a variety of important economic topics. We will find that some puzzling empirical findings (such as the decline of consumption at old age and the low share of equity holdings of young households) become a natural consequence of uncertain lifetime when we account for ICA. We also explain why introducing ICA leads to a fundamental revaluation of the role of mortality changes. In particular, the

link between mortality decline and economic development may be much stronger than is usually inferred from standard additive models. Moreover, considerations about the value of a life, and especially the relation between age and the value of a statistical life, are strongly affected by ICA.

The rest of this paper is structured as follows. In section 2, we give the notation. The main assumptions are given in Section 3. Under these assumptions, we show that lifetime preferences under uncertainty can be represented by recursive utility functions with finite horizons (Section 4). In Section 5, we discuss the fundamental properties of such preferences. This leads us to draw a typology of stationary preferences (Section 6). The relation between the fundamental concepts that we defined in Section 5 and what can be inferred from the observation of individuals who face non-degenerate mortality patterns is discussed in Section 7. We illustrate our discussion, in Section 8, by comparing the predictions of two particular kinds of utility functions, the additive and the multiplicative ones, when mortality is assumed to follow realistic patterns.

2 Notations

We assume that at any time an individual is either dead or consumes a single composite good. Consumption is assumed to take values in \mathbb{R}^+ . The death state will be denoted by the letter d . Thus, for any moment in time the state of an individual is an element of the set:

$$X = \mathbb{R}^+ \cup \{d\}$$

As the death state d is of a different nature than any consumption value, the set X does not have an obvious structure¹. The set X can, however, be endowed with a simple metric m defined by $m(x_1, x_2) = |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^+$,

¹Assuming that the death state corresponds to a zero or negative level of consumption may generate inelegant artificial continuity or monotonicity breakdowns in preferences.

$m(x, d) = +\infty$ for all $x \in \mathbb{R}^+$ and $m(d, d) = 0$ ².

A life is a function from \mathbb{R}^+ into X . For technical reasons, we only consider measurable functions. We note $F(\mathbb{R}^+, X)$ the set of such functions. Given that all individuals eventually die and that death is irreversible, the set of possible lives is defined by:

$$Z = \{z \in F(\mathbb{R}^+, X) \text{ such that the set of times } t \\ \text{with } z(t) \neq d \text{ is an interval of the form } [0, T_z], \text{ for some } T_z < \infty\}$$

For any $z \in Z$, we denote T_z the length of life and we define the consumption profile $c_z \in F([0, T_z], \mathbb{R}^+)$ by $c_z(t) = z(t)$ for any $t \in [0, T_z]$. A life z is fully characterized by the length of life T_z and by the consumption profile c_z . We will use indifferently the notation z or (c_z, T_z) to refer to a given life. The set Z is endowed with the weak topology³.

For any $z \in Z$, we denote δ_z the corresponding Dirac probability measure (i.e. the function from Z into \mathbb{R} such that $\delta_z(z) = 1$ and $\delta_z(z') = 0$ for any $z' \neq z$). By definition a lottery (i.e. a simple measure) on Z is a finite convex combination of Dirac probability measures. The set of lotteries on Z is denoted $L(Z)$. Elements of $L(Z)$ will either be denoted by simple letters (e.g. l) or by a sum of Dirac probability measures (e.g. $\sum_i \alpha_i \delta_{z_i}$). In this latter case it is always implicitly assumed that the sum is actually a finite convex combination. The set $L(Z)$ is endowed with the weak topology of measures⁴.

²Remark that the fact $m(x, d) = +\infty$ does not mean that two lives of different lengths cannot be close in topological terms. The set of lives, defined below, will be endowed with the weak topology. It is then possible that a sequence of lives of lengths $T_n \neq T$ converges towards a life of length T . Hence, agents with continuous preferences may admit trade-offs between length of life and consumption, even if $m(x, d) = +\infty$.

³A sequence $z_n \in Z$ converges to $z \in Z$ if and only if $m(z_n(t), z(t)) \rightarrow 0$ for almost every $t \in \mathbb{R}^+$.

⁴A sequence $l_n \in L(Z)$ converges to $l \in L(Z)$ if and only if

$$\sum_{z \in Z} l_n(z) f(z) \rightarrow \sum_{z \in Z} l(z) f(z)$$

for any continuous real valued function f on Z .

We sometimes refer to particular subsets of Z . In particular, we define:

$$Z_c = \{z \in Z \mid c_z \text{ is a continuous function}\}$$

and, for all $T > 0$,

$$Z_T = \{z \in Z \mid T_z = T\}$$

we denote $L(Z_c)$ and $L(Z_T)$ the set of lotteries on Z_c and Z_T .

For any $t_0 \in \mathbb{R}^+$ any $c_0 \in F([0, t_0], \mathbb{R}^+)$ and any $z \in Z$, we define $c_0 *_{t_0} z \in Z$ by:

$$\begin{aligned} c_0 *_{t_0} z(t) &= c_0(t) \text{ if } t < t_0 \\ c_0 *_{t_0} z(t) &= z(t - t_0) \text{ if } t \geq t_0 \end{aligned}$$

The life $c_0 *_{t_0} z$ is the life that follows the consumption path c_0 until t_0 and then follows the life z , starting from $z(0)$.

For any lottery $l = \sum_i \alpha_i \delta_{z_i} \in L(Z)$ the lottery $c_0 *_{t_0} l \in L(Z)$ is then defined by:

$$c_0 *_{t_0} l = \sum_i \alpha_i \delta_{c_0 *_{t_0} z_i}$$

The lottery $c_0 *_{t_0} l$ is therefore the lottery in which individuals have the consumption path c_0 during the first t_0 years of their life and then live and die according to the lottery l .

3 Assumptions

We now state the axioms that will lead to our representation result.

Axiom 1 (*Ordering*) *Individuals have a rational preference relation (i.e. a complete preorder) on $L(Z)$.*

We denote by \succeq the preference relation, \succ the strict preference relation, and \sim the indifference relation.

Axiom 2 (*Independence*) For any $l_1, l_2, l_3 \in L(Z)$ and any $\lambda \in (0, 1)$

$$l_1 \succ l_2 \implies \lambda l_1 + (1 - \lambda)l_3 \succ \lambda l_2 + (1 - \lambda)l_3$$

Axiom 3 (*Continuity*) For any $l_1 \in L(Z)$ the sets

$$\{l \in L(Z) | l \succ l_1\} \text{ and } \{l \in L(Z) | l_1 \succ l\}$$

are open.

The first two axioms are usual axioms of the expected utility theory. The third axiom is a standard expression of the continuity of preferences. However, this axiom is stronger than the continuity condition found in standard economic textbooks⁵ which relies on a different topology for $L(Z)$ ⁶.

The fourth axiom is a technical assumption of non-satiation:

Axiom 4 (*Non-satiation*) For any $T > 0$ and any $z \in Z_c \cap Z_T$ there exists $z_1 \in Z_c \cap Z_T$ such that

$$\delta_{z_1} \succ \delta_z$$

The last axiom expresses the assumption of stationarity.

Axiom 5 (*Stationarity*) For any $t_0 \in \mathbb{R}^+$ any $c^0 \in F([0, t_0], \mathbb{R}^+)$ and any $l, l' \in L(Z)$ we have:

$$l \succ l' \Leftrightarrow (c_0 *_{t_0} l) \succ (c_0 *_{t_0} l')$$

The assumption of stationarity implies that preferences are history independent and time consistent. Therefore, two individuals of different ages, say a 30

⁵See for example Mas-Colell, Whinston and Green (1995), page 171.

⁶Continuity of preferences may also be formulated when $L(Z)$ is endowed with the following metric:

$$M(l_1, l_2) = \sum_{z \in Z} |l_1(z) - l_2(z)|$$

The corresponding continuity condition is weaker than our third axiom but suffices to guarantee (together with Axioms 1 and 2) that preferences can be represented by a von Neumann-Morgenstern utility functions. Our stronger notion of continuity (that is based on the weak topology defined in footnote 4) makes it possible to navigate between the discrete and continuous time frameworks in the proof of Theorem 1.

year old and 60 year old, are assumed to have the same preferences. That does not imply that they will behave in the same way. They would do so only if they faced the same constraints: that is the same budget constraints and the same mortality risks. In practice mortality strongly depends on age, and a 30 year old and a 60 year old are confronted with radically different constraints. Consequently, we expect them to behave very differently, even if they have the same preferences (see for example the illustrations provided in Section 8). Stated otherwise, stationarity involves assuming that age is a relevant variable only because it affects individual constraints (in particular those related to mortality). Such an assumption can of course be turned into ridiculous by looking at extreme cases. A baby who only has a few years of life expectancy because of an incurable disease does not behave as 95 year old individual with similar mortality rates⁷. In other words, age may also matter through other channels than mortality. But to the extent that life cycle theory aims at providing economic insights on why individual behavior changes along the life cycle, the stationarity assumption seems to be the most reasonable choice, at least for a starting point⁸.

An important feature of our axiomatic formulation is that it applies the von Neumann-Morgenstern to the set of atemporal lotteries. Therefore, we remain in the standard expected utility theory and do not follow the direction initiated by Kreps and Porteus (1978) that applies the von Neumann-Morgenstern framework to temporal lotteries in order to obtain more general models of dynamic choice. Actually, Corollary 3 of Kreps and Porteus (1978) tells us that our framework can be considered as a particular case of Kreps and Porteus's dynamic choice theory where individuals are indifferent to the timing of resolution of uncertainty. Whether or not such an assumption of indifference should be relaxed is open to debate. Dynamic choice theory indisputably offers a greater flexibility. But it is also much more complex than expected utility theory. In fact, most papers that

⁷Still one may argue that the problem is not with the stationarity assumption, but with the fact that some age-specific constraints (physical ability, etc.) are not introduced in the model.

⁸It might be tempting to introduce a relation between age and preferences. But that would open the door to *ad-hoc* assumptions that would *in fine* limit the explanatory power of the theory.

use dynamic choice theory respond to this increase in complexity by assuming particular specifications. Instead, we prefer to remain in the simpler framework provided by the expected utility theory but consider all the specifications that are consistent with the above axioms.

4 The set of stationary preferences

It is well known that Axioms 1 to 3 imply that preferences on $L(Z)$ can be represented by a von Neumann-Morgenstern utility function. Our first result consists in showing that when preferences are stationary, the von Neumann-Morgenstern utility function has a simple expression:

Theorem 1 *If Axioms 1 to 5 are fulfilled then there exist two functions u and v from \mathbb{R}^+ into \mathbb{R} such that the relation of preferences on $L(Z_c)$ can be represented by the von Neumann-Morgenstern utility function:*

$$\left\{ \begin{array}{l} Z_c \rightarrow \mathbb{R} \\ z \rightarrow U(z) = U(c_z, T_z) = \int_0^{T_z} u(c_z(t)) \exp \left(- \int_0^t v(c_z(\tau)) d\tau \right) dt \end{array} \right. \quad (1)$$

Reciprocally, preferences represented by a von Neumann-Morgenstern of utility function of the form given in (1), with continuous functions u and v and an increasing function u , fulfill Axioms 1 to 5.

Proof. In Appendix A. ■

A representation result that looks similar has been provided (in the discrete time case) by Epstein (1983). The key difference between Epstein (1983) and our contribution is that Epstein deals with infinitely long lived agents. Consequently, Epstein only obtains representations where agents have pure time preferences. Instead, our framework which accounts for the finiteness of human life makes it possible to consider the case where agents do not have pure time preferences. This is particularly important, since the model with no time preferences, far from being a curiosity, will prove to be a very serious alternative to the model of Yaari (see the discussion in Section 8).

In the set of utility functions of the form (1), there are two particular subsets that will be of special importance throughout the paper. We give them a formal name:

Definition 1 *Preferences will be called “additive” if they can be represented by a von Neumann-Morgenstern utility function of the form (1) with the function $v = \beta$ constant. In such a case (1) rewrites:*

$$U(c_z, T_z) = \int_0^{T_z} u(c_z(t))e^{-\beta t} dt \quad (2)$$

Definition 2 *Preferences will be said to be “multiplicative” if they can be represented by a von Neumann-Morgenstern utility function of the form (1) with a function $v = ku$ for some constant k . In such a case, integration of (1) leads to:*

$$\begin{aligned} U(c_z, T_z) &= \frac{1}{k} \left(1 - \exp \left(-k \int_0^{T_z} u(c_z(\tau)) d\tau \right) \right) \text{ if } k \neq 0 \\ U(c_z, T_z) &= \int_0^{T_z} u(c_z(t)) dt \text{ if } k = 0 \end{aligned} \quad (3)$$

It is worth mentioning that although the additive formulation is the most common in the economic literature, there have been several theoretical papers on intertemporal choice with a finite exogenous horizon that have advocated the multiplicative form (see Richard, 1975, for example). However, this multiplicative form has most frequently been left aside. One of the main reasons for this is that when T tends to infinity, equation (3) does not always converge to a finite limit. Thus, multiplicative preferences appear then to be inappropriate to define preferences over the set of infinitely long lives and, as such, do not appear in Epstein (1983)⁹. The question of convergence, however, is no longer problematic when comparing life streams that remain in a particular state (death in the present case) after a finite time.

⁹There are several papers that avoid this problem by adding an exogenous rate of discount in equation (3) (see for example Pye, 1973, or Ahn, 1989). Equation (3) then becomes $U(c, T) = \frac{1}{k} \left(1 - \exp \left(-k \int_0^T e^{-\beta t} u(c(\tau)) d\tau \right) \right)$. But, as pointed at by Epstein (1992), when β and k are different from zero such preferences are non stationary

5 Properties of stationary von Neumann-Morgenstern utility functions

The general form of the von Neumann-Morgenstern utility functions that represent stationary preferences is given in equation (1). However, the meanings of the functions u and v that appear in this formulation are rather unclear. For example, we know that when v is constant it represents the rate of time preference. But it is not clear what the rate of time preference is when v is not constant. Similarly, it is not obvious, a priori, to interpret the derivative of v , etc. This section provides general definitions of intuitive concepts of intertemporal choice theory and deduces what their expressions turn out to be when the utility function has the form given in (1). The meanings of u and v will then become clearer. Some particular specifications will also appear as corresponding to fundamental assumptions.

This section, as well as the remainder of the paper, will make use of Volterra derivatives that make it possible to define in a simple way standard economic concepts in the case of an individual who cares for a continuum of goods (e.g. consumption at each age in a continuous time model)¹⁰. In order to avoid technical difficulties we make two additional assumptions.

Assumption 1 *The functions u and v that appear in equation (1) are twice continuously differentiable.*

Assumption 2 *The functions u and v are such that for any life (c, T) and any age $t < T$, we have:*

$$\frac{\partial U(c, T)}{\partial c(t)} > 0 \quad .$$

Rather than reviewing all the standard concepts of intertemporal choice theory (risk aversion, intertemporal elasticity of substitution, etc.) we will focus on two concepts that are central in our discussion: time preference, on the one hand, and ICA on the other hand.

¹⁰See Ryder and Heal (1973) for a former use of Volterra derivatives in economics.

Time preference is a familiar notion, usually measured by the rate of time preference. We follow Epstein (1987) in defining the **rate of time preference** at time t by:

$$RTP_t = -\frac{d}{dt}(\log \frac{\partial U(c, T)}{\partial c(t)})|_{c'(t)=0} \quad . \quad (4)$$

The rate of time preference simply describes how the marginal utility of consumption varies along the life cycle when controlling for the variations in consumption. Simple derivations lead to:

$$RTP_t = \frac{v(c(t))u'(c(t)) - v'(c(t))u(c(t))}{u'(c(t)) - v'(c(t)) \int_t^T u(c(\tau_1)) \exp(-\int_t^{\tau_1} v(c(\tau))d\tau) d\tau_1} \quad (5)$$

The notion of ICA is much less well known, although it occasionally appeared in the economic literature¹¹. We use the three words “intertemporal correlation aversion” to stress that it corresponds to a particular measure of correlation aversion that can be defined in the intertemporal framework.

Correlation aversion, which itself is not very well known, is a natural concept when looking at preferences over several attributes under uncertainty. It has been separately introduced, under different names, by de Finetti (1952) and Richard (1975). The terminology “correlation aversion” comes from Epstein and Tanny (1980). Basically, when considering preferences over bivariate lotteries, correlation aversion tells whether an individual prefers lotteries that exhibit a positive or a negative correlation. Formally, preferences over bivariate lotteries exhibit a positive correlation aversion if and only if, for all x_1, x_2, y_1, y_2 , such that $x_1 < x_2$ and $y_1 < y_2$, the lottery:

$$\left\{ \begin{array}{ll} (x_1, y_2) & \text{w.p. } \frac{1}{2} \\ (x_2, y_1) & \text{w.p. } \frac{1}{2} \end{array} \right\} \text{ is preferred to the lottery } \left\{ \begin{array}{ll} (x_1, y_1) & \text{w.p. } \frac{1}{2} \\ (x_2, y_2) & \text{w.p. } \frac{1}{2} \end{array} \right\}$$

In the intertemporal framework, Bommier (2003) suggests measuring correlation aversion with respect to consumption at age t_0 and consumption at age t_1 by the

¹¹See Ahn (1989), for example.

following index:

$$\rho_{t_0, t_1} = -2 \frac{\frac{\partial^2 U(c, T)}{\partial c(t_0) \partial c(t_1)}}{\frac{\partial U(c, T)}{\partial c(t_0)} + \frac{\partial U(c, T)}{\partial c(t_1)}}$$

It is positive (resp. negative) if individuals prefer consumption lotteries at times t_0 and t_1 to be negatively (resp. correlated) correlated. It is then explained that this index of correlation aversion can be related to the amount of consumption that is necessary to compensate for a positive and infinitesimally small correlation. A local measure of correlation aversion can then be defined by taking the limit of ρ_{t_0, t_1} when $t_1 \rightarrow t_0$:

Definition 3 *For any length of life T and any consumption profile, c , the **intertemporal correlation aversion** at time $t < T$ is defined by:*

$$\rho_t = - \lim_{\varepsilon \rightarrow 0, \varepsilon \neq 0} \frac{\frac{\partial^2 U(c, T)}{\partial c(t+\varepsilon) \partial c(t)}}{\frac{\partial U(c, T)}{\partial c(t)}} \quad (6)$$

The expression that relates ICA to the functions u and v is very simple:

$$\rho_t = v'(c(t)) \quad (7)$$

ICA proves to be an important characteristic of individual preferences in a number of instances. For example, Bommier (2003) shows that ICA is simply related to the difference between local measures of relative risk aversion and intertemporal elasticity of substitution. It is thus natural to see ICA playing a major role for determining optimal financial strategies, as shown in Bommier and Rochet (2005).

Still, ICA is probably even more important when accounting for the risk of death. The point is that death is an irreversible event. Thus in the case of death at age t , the individual states at ages greater than t are all set to the “death” state. In other words, the risk of death is akin to a sequence of correlated risk on individual’s future states. Consequently, we expect ICA to strongly affect how individuals respond to the risk of death.

In some cases, the role of ICA will be easier to explain when considering the notion of risk aversion with respect to length of life that we introduce below. To begin with we define the **gross risk aversion with respect to length of life**:

$$GRAL_T = - \frac{\frac{\partial^2 U(c,T)}{\partial T^2}}{\frac{\partial U(c,T)}{\partial T}}$$

The gross risk aversion with respect to length of life is simply an Arrow-Pratt coefficient of absolute risk aversion in the case where the variable of interest is the length of life. Thus, it determines how an individual with an exogenous consumption profile would rank lotteries on the length of life. Although this concept is particularly important for medical decision-making¹², it has rarely been discussed in the economic literature¹³. It is simple to see that:

$$GRAL_T = v(c(T)) - c'(T) \frac{u'(c(T))}{u(c(T))} \quad (8)$$

This index of risk aversion is labeled “gross” to emphasize that it actually results from several factors that we may want to disentangle for a theoretical discussion. First, it is clear from (8) that it depends on the shape of the consumption profile. In the usual case where individuals would prefer to live longer ($u(c(T)) > 0$) gross risk aversion with respect to length of life is greater if the consumption profile is decreasing than if it is constant. Clearly, if life at old age brings few pleasures compared to life at younger ages, there is little incentive to take the risk of dying young (for example by undergoing a preventive surgical operation) in order to increase the probability of living to an old age. Second, preferences over lotteries on length of life are also affected by time preferences. If utility is highly discounted in the future, individuals are strongly risk averse with respect to length of life as they do not want to take a chance on short term survival

¹²It was first introduced in the seminal paper of McNeil, Weichselbaum and Pauker (1978) who discussed the appropriateness of various risky treatments for lung cancer.

¹³We found it only in the few papers that intend to depart from the additive formulation. It appears for example, under the name “utility curvature”, in Bleichrodt, van Rijn and Johannesson (1999).

for a gain on a highly discounted period of life. In order to obtain a concept of risk aversion with respect to length of life which is net of the variations in consumption and net of time preferences, we propose the following concept:

Definition 4 *For any length of life T and any consumption profile c , the **(net) risk aversion with respect to length of life** is defined by:*

$$RAL_T = -\frac{d}{dT} \log \left(\frac{\partial U(c, T)}{\partial T} \right) \Big|_{c'(T)=0} + \frac{d}{dt} \log \left(\frac{\partial U(c, T)}{\partial c(t)} \right) \Big|_{t=T, c'(T)=0} \quad (9)$$

The net risk aversion with respect to length of life therefore tells how the marginal utility of life decreases with age, relative to the marginal utility of consumption, when controlling for variations in consumption. Net risk aversion with respect to length of life is given by:

$$RAL_T = \frac{v'(c(T))u(c(T))}{u'(c(T))} \quad (10)$$

Thus:

$$RAL_T = \rho_t \frac{\frac{\partial U(c, T)}{\partial T}}{\frac{\partial U(c, T)}{\partial c(t)} \Big|_{t=T}}$$

and the net risk aversion with respect to length of life equals the product of ICA by the marginal rate of substitution between length of life and consumption near the end of life. In particular, the standard additive model, which assumes a zero ICA, also assumes that individuals are risk neutral with respect to length of life. In the usual case where individuals would prefer to live longer ($u(c(t)) > 0$), individuals with positive ICA are risk averse with respect to length of life and vice versa.

6 A typology of stationary lifetime preferences

Within the set of stationary preferences there are several subsets that are characterized by remarkable properties. Let us first express very simple results that essentially summarize in formal terms what appear from the analytic expressions

that were given in the previous section:

Proposition 1 *The following statements are equivalent:*

1. *Preferences are additive or multiplicative.*
2. *Preferences are ordinally interindependent (in the sense of Ryder and Heal, 1973)¹⁴.*
3. *The rate of time preference is independent of the consumption profile, age and the length of life.*
4. *The rate of time preference is constant along any constant consumption path.*

Proof. See Appendix B. ■

Proposition 2 *The following statements are equivalent:*

1. *Preferences are additive.*
2. *Intertemporal correlation aversion always equals zero.*

Proof. It follows immediately from equation (7). ■

Proposition 3 *The following statements are equivalent:*

1. *Preferences are multiplicative.*
2. *The rate of time preference always equals zero.*

¹⁴Ordinal interindependence means that the marginal rate of substitution between consumptions at two different ages is unaffected by consumption at another age. In other words, preferences are ordinally interindependent if and only if

$$\frac{\partial}{\partial c(t_1)} \frac{\frac{\partial U(c,T)}{\partial c(t_2)}}{\frac{\partial U(c,T)}{\partial c(t_3)}} = 0$$

for any distinct t_1, t_2 and t_3 .

Proof. From (5), the rate of time preference always equals zero if and only if $u(c(t))v'(c(t)) = u'(c(t))v(c(t))$ for all $c(t)$. Under assumption 1, this is the case if and only if u and v are proportional (which, by definition, means that preferences are multiplicative). ■

A simple picture of the set of stationary preferences follows from these three propositions (see Figure 1).

At this point, however, we must be careful not to misuse our intuition to rule out some particular specifications. For example, stationary preferences can be represented by a multiplicative utility function if and only if individuals have no pure time preferences. This may seem an unpleasant assumption and in contradiction with empirical findings indicating that individuals prefer present consumption over future consumption. However, it is worth stressing that in the presence of uncertainty, ICA can generate sizable time discounting. In particular, when individuals face an exogenous uncertain lifetime, it can be the case that ICA generates discount rates that are comparable in size with what it is usually assumed in the economics literature. This is explained in the following section.

7 Discount rate and intertemporal correlation aversion with non-degenerate random mortality

In reality, there is always a considerable uncertainty about the length of life, *ex ante*. Thus, the concepts that are defined for a given length of life (such as those discussed in Section 5) are not directly observed. Typically, what can be observed are marginal concepts that describe individual preferences in a neighborhood of non-degenerate lotteries on the length of life. For example, we do not observe individuals' rate of time preference, but how individuals discount consumption knowing that the length of life is uncertain. Also, we do not observe the marginal rate of substitution between length of life and consumption, but the willingness

to pay for reducing the hazard risk of death at a given moment in time. The aim of this section is to make explicit the link between time preferences, ICA and what can be inferred from the behavior of mortal human beings.

In the following, a mortality pattern, that we will denote by the letter μ , will be described either by the hazard rate of death function $\mu(t)$, by the distribution of the age at death $d_\mu(T) = \mu(T) \exp(-\int_0^T \mu(t)dt)$ or by the survival function $s_\mu(t) = \exp(-\int_0^t \mu(\tau)d\tau)$. It is assumed that $\mu(t) \rightarrow +\infty$ when $t \rightarrow +\infty$.

For any function $f(x, T)$ that depends on some attributes, x , and on the age at death, T , we define the μ, t -average of $f(x, T)$ that we denote $E_{\mu,t}f(x, \cdot)$, by:

$$E_{\mu,t}f(x, \cdot) = \frac{\int_t^{+\infty} d_\mu(T) f(x, T) dT}{\int_t^{+\infty} d_\mu(T) dT} \quad (11)$$

$E_{\mu,t}f(x, \cdot)$ is simply the average of $f(x, T)$ when T follows the distribution of the age at death truncated at $T \geq t$. For example, in the case where $f(x, T)$ is the age at death (that is when $f(x, T) = T$), then $E_{\mu,t}f(x, \cdot)$ is the average age at death of the individuals that are still alive at age t .

Remark that for any continuously differentiable function f , an integration by parts of (11) leads to:

$$E_{\mu,t}f(x, \cdot) = f(x, t) + \int_t^{+\infty} \frac{s_\mu(\tau)}{s_\mu(t)} \left(\frac{\partial}{\partial T} f(x, T) \right)_{|T=\tau} d\tau \quad (12)$$

so that the μ, t -average of $f(x, T)$ can be expressed as a function of the survival function, instead of a function of the distribution of the age at death. We will use indifferently equation (11) or equation (12) in the remainder of the paper.

Rational individuals with a von Neumann-Morgenstern utility function (1) aim at maximizing the expected utility:

$$E_{\mu,0}U(c, \cdot) = \int_0^{+\infty} d_\mu(T) \int_0^T u(c(t)) \exp\left(-\int_0^t v(c(\tau))d\tau\right) dt dT$$

Using (12), this can be rewritten as:

$$E_{\mu,0}U(c, \cdot) = \int_0^{+\infty} s_{\mu}(t)u(c(t)) \exp\left(-\int_0^t v(c(\tau))d\tau\right) dt \quad (13)$$

We recognize here a generalization of the lifetime utility function suggested by Yaari (1965), which is obtained when v is constant. Local properties of individual preferences in the neighborhood of elements of $L(Z)$ characterized by a given consumption profile and a non degenerate mortality pattern can be expressed using Volterra derivatives of (13). In particular:

Definition 5 *For any mortality pattern and any consumption profile, we define the μ -rate of discount at time t by:*

$$RTP_{\mu,t} = -\frac{d}{dt}(\log \frac{\partial E_{\mu,0}U(c, \cdot)}{\partial c(t)})|_{c'(t)=0}$$

Definition 6 *For any mortality pattern and any consumption profile, we define the μ -intertemporal correlation aversion at time t by:*

$$\rho_{\mu,t} = -\lim_{\varepsilon \rightarrow 0, \varepsilon \neq 0} \frac{\frac{\partial^2 E_{\mu,0}U(c, \cdot)}{\partial c(t) \partial c(t+\varepsilon)}}{\frac{\partial E_{\mu,0}U(c, \cdot)}{\partial c(t)}}$$

Definition 7 *For any mortality pattern and any consumption profile, the Value of Statistical Life at age t is:*

$$VSL_{\mu,t} = -\frac{\frac{\partial E_{\mu,0}U(c, \cdot)}{\partial \mu(t)}}{\frac{\partial E_{\mu,0}U(c, \cdot)}{\partial c(t)}}$$

Definitions 5 and 6 extend the definitions given by (4) and (6) to the case where the length of life is not known with certainty. The Value of a Statistical Life, $VSL_{\mu,t}$, is nothing else than the opposite of the marginal rate of substitution between mortality at time t and consumption at time t . In practice for an infinitesimally small value $d\mu$ and an infinitesimally small lapse of time dt , $VSL_{\mu,t}d\mu$ gives the level of consumption that an individual is willing to give up

during dt periods of time around t in order to reduce his hazard rate of death from $\mu(\tau)$ to $\mu(\tau) - d\mu$ during dt periods of time around t . The terminology Value of Statistical Life is consistent with that used in Johansson (2002).

We can now express the following result:

Theorem 2 *For any consumption profile, c , any mortality pattern, μ , and any time, t , we have*

$$\rho_{\mu,t} = \rho_t \quad (14)$$

$$RTP_{\mu,t} = \frac{1}{E_{\mu,t}(\frac{1}{RTP_t})} + \mu(t) + \mu(t)\rho_t VSL_{\mu,t} \quad (15)$$

Proof. See Appendix C. ■

We see therefore that the μ -ICA is simply equal to the ICA. Therefore, the observation of ICA is not complicated by the presence of mortality. The same statement is not true however for the rate of time preference.

We can observe from equation (15) that the μ -rate of discount is the sum of three terms. The first one, $\frac{1}{E_{\mu,t}(\frac{1}{RTP_t})}$, is the harmonic mean of the rate of time preference. This term accounts for individuals' pure time preferences. In the case of the additive or multiplicative model, the rate of time preference is a constant and the first term is simply the exogenous rate of time preference. However, in the general case, the rate of time preference may depend on the length of life. Equation (15) indicates that it is its harmonic mean that matters when the length of life is random.

The second term is the mortality rate. It accounts for the fact that mortality creates a risk on future consumption. Consumption only occurs in case of survival.

The third term stresses the role of ICA. It vanishes when ICA equals zero, and consequently has remained unnoticed in the economic literature that relies on Yaari's model to account for lifetime uncertainty.

Two parallel lines of argument can be followed to provide intuition about the origin of this third term. The first one sticks to the crude meaning of ICA. Mortality generates a risk on tomorrow's consumption that is positively correlated

to a much greater risk: the risk of losing life for ever. If tomorrow's consumption proves to be impossible, because of death, so will be the case of consumption at any date after tomorrow. Agents with positive ICA will react to this correlation by decreasing the risk on tomorrow's consumption: that is by consuming more today and less tomorrow. The magnitude of the reaction depends on the mortality risk, ICA and the value of the other items at risk (life, whose value is given by $VSL_{\mu,t}$). That explains the structure of this third term.

Another way to think about the third term of (15) involves making the link with the notion of risk aversion with respect to length of life. Mortality makes lifetime utility random: lifetime utility is low in case of an early death, and high in case of a late death. However, reallocating of consumption towards early periods of life is a way to make the distribution of lifetime utility less unequal. By consuming early in the life cycle, one increases lifetime utility of short lives and lowers the utility gap between short and long lives. The willingness to reallocate consumption for that purpose obviously depends on individuals' risk aversion. It vanishes when individuals are risk neutral with respect to length of life (that is in the additive model) and increases with risk aversion with respect to length of life.

It is important to note that this third term can generate substantial time discounting, even if mortality and ICA are small, since the Value of a Statistical life is usually estimated to be very large.

If time discounting may be only partially driven by time preferences, the key question is how we can identify, at least in theory, individuals' pure time preferences. Theorem 2 provides a solution. Indeed, from equation (15), we know that time discounting at age t is a linear function of the mortality rate at age t , with slope $(1 + \rho_t VSL_{\mu,t})$ and intercept $\frac{1}{E_{\mu,t}(\frac{1}{RTP_t})}$. Thus, in absence of pure time preferences the elasticity of the μ -rate of discount at age t with respect to mortality at age t equals 1. If individuals have positive pure time preferences then this elasticity is smaller than one. More generally, a regression of the rate of discount at age t with a list of variables including mortality rate at

age t could theoretically provide estimates of both $E_{\mu,t}(\frac{1}{RTP_t})$ and $\rho_t VSL_{\mu,t}$. The difficulty of the task should not be underestimated, however, since one needs to control for mortality rates at ages greater than t , which are strongly correlated with the mortality rate at age t . Following this direction would thus require to have simultaneous estimates on individual rates of discount, on the one hand, and mortality rates at all (present and future) ages, on the other hand.

From equation (14), we know that identifying individual ICA should be less problematic, since $\rho_{\mu,t}$ and ρ_t are equal. Experiments measuring how human (and mortal) beings would rank intertemporal lotteries that are more or less correlated could provide a direct estimate of $\rho_{\mu,t}$. Moreover, as ICA is linked to risk aversion with respect to length of life, another possibility is to look at endogenous mortality choices. This latter approach is followed in Bommier and Villeneuve (2004).

8 Additive and multiplicative utility functions

In order to stress further the role of ICA, in this section we discuss how the additive and the multiplicative models compare when applied to individuals whose age-specific mortality rates conform with what is reported by demographic studies¹⁵. The whole section is based on numerical simulations.

The additive and multiplicative models can appear as two extensions of the simplest model where

$$U(c, T) = \int_0^T u(c(t))dt$$

which assumes both a zero ICA and a zero rate of time preference. The additive model allows for time preferences but maintains the assumption of zero ICA while the multiplicative model allows for ICA but maintains the assumption of a zero rate of time preference. Thus, by comparing the predictions of the additive and

¹⁵Demographic data is taken from the Berkeley Mortality Database which provides gender-, country- and time-specific mortality rates. Figures 2, 3, 4 and 8 are drawn using the 2000 (male-female average) US mortality rates. Figure 5, which focuses on gender differences, uses the 2000 gender-specific US mortality rates. Figures 6 and 7 which explore the impact of mortality changes is based on the historical and projected (male-female average) US mortality rates.

multiplicative models, we get a first idea of what life-cycle theory might look like if the paradigm of time preferences were abandoned for that of ICA.

Although very different on pure theoretical grounds, the additive and multiplicative models may lead to comparable predictions on some issues. Look for example at an individual's rate of discount. From (15), we know that the rate of discount, is the sum of (i) the rate of time preference, (ii) the mortality rate and (iii) a term driven by ICA. The second component (the mortality rate) being a purely demographic factor, the differences between the multiplicative and the additive models are found in the first and third terms. If we compare them, term by term, the contrast between the additive and the multiplicative model is unambiguous. However, when we add all the three terms together it is not obvious that the additive and multiplicative model will contrast so distinctly.

As an illustration, we plot on Figure 2 the rate of discount along a constant consumption path. In the additive model, we assume that the rate of time preference equals 0.030 per year, while for the multiplicative model, we assume that risk aversion with respect to length of life equals 0.088 per year¹⁶. At first glance, the difference between the two models is not huge. In some ways, that is reassuring, since it seems to indicate that the choice we may have to make between the additive and multiplicative models is not crucial.

The apparent proximity in the rates of discount that we can see in Figure 2 hides nonetheless sizable differences. In the additive model the difference between the rate of discount and the mortality rate is a constant. As a consequence, if credit and annuity markets are perfect, the optimal consumption path is monotonic with age. This is not true with the multiplicative model. This is illustrated in Figure 3 where we plot optimal consumption paths predicted by the additive

¹⁶The value of 0.030 per year for the time preference parameter is a standard choice in the economic literature. That of 0.088 per year for the risk aversion with respect to length of life was chosen so that both models give similar predictions with the mortality observed in year 2000. With such a value, a 20 year old individual, with a background mortality given by the 2000 US life table, would be indifferent between going through a surgical intervention that he will survive with probability 0.90 or having to face an extra risk of death of 0.01 per year during all the remainder of her or his life.

and multiplicative models¹⁷. Income is supposed to be exogenous and certain. It equals 1 before age 65 and 0.5 for ages 60 and older¹⁸. The rate of interest is supposed to be exogenous and equals 0.035 per year. We find that the multiplicative model predicts a decline in consumption after age 65, while consumption would keep on rising according to the additive model. This fall in consumption is interesting as it shares some similarity with what has been observed in practice and often presented as puzzling (Hamermesh, 1984)¹⁹.

The additive and multiplicative models also have rather different cardinal properties. This turns out to be important when we look at individuals' risk aversion. Consider for example the relative risk aversion with respect to remaining lifetime income. In a most standard way, we define it by:

$$R_{\mu,t} = -\frac{WV''_{\mu,t}(W_t)}{V'_{\mu,t}(W_t)}$$

where $W_t = \int_t^{+\infty} \frac{s_\mu(\tau)}{s_\mu(t)} y(\tau) e^{-r(\tau-t)} d\tau$ is the expected remaining lifetime income at

¹⁷ As above, we assume that the rate of time preference equals 0.030 in the additive model and that risk aversion with respect to length of life equals 0.088 in the multiplicative model when $c = 1$. We also assume, here and in the remainder of the paper, that, in both models, the intertemporal elasticity of substitution is constant and equals 1.5. Thus, in the additive model we have

$$U(c, T) = \int_0^T (c_a^{-0.5} - c(t)^{-0.5}) e^{-0.03t} dt$$

while in the multiplicative model:

$$U(c, T) = -\exp(-0.88 \int_0^T (\frac{c_m^{-0.5} - c(t)^{-0.5}}{c_m^{-0.5} - 1}) dt)$$

The constants c_a and c_m were chosen to be consistent with standard estimates of the value of a statistical life. More precisely, we chose c_a and c_m so that an individual having a background mortality provided by the 2000 US life table and an income of 1 would be indifferent between facing an extra risk of death of 0.0001 per year, from age 20 till the end of his/her life, or having an income of 1.035. That is consistent with a (survival weighted) average value of a statistical life of 7 million dollars (a reasonable value, according to Viscusi and Aldy, 2003) if we assume that one unit of income actually corresponds to 20000 dollars per year.

¹⁸At this point, the shape of the income profile does not matter. However, this is no longer the case when we look at aggregate savings (which is necessary for Figure 7).

¹⁹Note that even if the multiplicative model provides a possible explanation for the decline of consumption at old ages, there are also many others plausible explanations that can be suggested as, for example, in Banks, Blundell and Tanner (1998).

age t and

$$V_{\mu,t}(W) = \max_c E_{\mu,t}(U(c, \cdot)) \text{ s.t. } \int_t^{+\infty} \frac{s_{\mu}(\tau)}{s_{\mu}(t)} c(\tau) e^{-r(\tau-t)} d\tau = W_t$$

is the indirect utility function at age t (we assume that financial markets are perfect). In the additive model, $R_{\mu,t}$ can be expressed analytically. When preferences exhibit a constant intertemporal elasticity of substitution, σ , (as in our simulation) then $R_{\mu,t}$ is independent of t and equals $\frac{1}{\sigma}$. This results from the inability of the additive model to separate risk aversion and intertemporal elasticity of substitution. With the multiplicative model we could not derive an analytic expression of $R_{\mu,t}$. However, numerical estimations can be provided (see Figure 4). We find that $R_{\mu,t}$ is always greater than $\frac{1}{\sigma}$ and decreases with age. Although we assume the intertemporal elasticity of substitution is the same in both models, agents appear as being more risk averse according to the multiplicative model than to the additive model. The gap between the prediction of the two models is greater for younger individuals (who have a longer horizon) than for older ones²⁰. The multiplicative model helps to explain why households, and especially young households, are found to hold little risky assets (see, for example, Ameriks and Zeldes, 2001 or Iwaisako, 2003).

An even more striking difference between additive and multiplicative models, is that they do not have the same sensitivities to mortality patterns. This can be illustrated by considering the consequences of exogenous heterogeneity in mortality. Consider, for example, the contrast between male and female mortality and see how it translates into rates of discount when we assume that men and women have the same preferences (Figure 5). According to the additive model, the difference between male and female discount rates is exactly given by the difference in their mortality rates. At age 30 the difference between male and female rates of discount would then be of 0.0011 per year. The difference is positive, but tiny, if we compare to values that are usually taken for the rate of time

²⁰Further intuition and theoretical results on the relation between horizon length and relative risk aversion are provided in Bommier and Rochet (2005).

preference. According to the multiplicative model the difference can be much larger since the third term in (15) also depends on mortality. Actually, with the parameters we consider, the difference in the rate of discount at age 30 is 0.012. That is 11 times larger than what the additive model predicts. Men would appear significantly more impatient than women. Similar simulations using data on differential mortality by education or wealth can also be done, showing that the multiplicative model predicts sizable heterogeneity in the rates of discount, while the additive model is only able to predict very small differences. Thus, the multiplicative model appears a much better candidate than the additive one to formalize Fisher’s intuition, that the “shortness of life tends powerfully to increase the degree of impatience, or rate of time preference, beyond what it would otherwise be” (Fisher 1930, p. 85).

Another way to illustrate the role of mortality is to look at the historical decline in mortality. In Figure 6, we plot the predicted rate of discount at age 50 for individuals who live according to mortality patterns taken from historical and projected US life-tables. For the additive model, the curve shown in Figure 6 exactly follows the evolution of the yearly mortality rate of a 50-year-old individual. Between 1900 and 2000 this rate goes down by 0.0109 and still decreases by 0.0019 between 2000 and 2080. According to the multiplicative model, the decline in mortality has a much greater effect. Between 1900 and 2000 the rate of discount loses 0.0298 and still loses 0.0089 between 2000 and 2080. Between 1900 and 2080 the estimated impact of the decline in mortality is 3.0 times greater in the multiplicative model than in the additive model. Given the sensitivity of economic predictions on savings, human capital investment and economic growth to discount rate values, this difference is anything but negligible.

To provide an order of magnitude we compute how the rate of interest would have changed over time in a simple economy, if individuals’ preferences and production functions had remained constant over time, and if the only exogenous element to change over time was mortality. More precisely, we do the following comparative static exercise. For each year between 1900 and 2080 we compute

the rate of interest that we would observe in a steady state equilibrium of an economy where (i) mortality is given by the (historical or projected) US life table of that year (ii) population growth rate equals zero (iii) the production function is given by a Cobb-Douglas function of the form $F(K, L) = AK^{0.3}L^{0.7}$ where K and L represent aggregate capital and labor in the economy²¹ (iv) individuals start working and consuming at age 20, (v) the age-specific labor income profile follows, up to a multiplicative constant (that equals 1 in year 2000), the path shown in Figure 3, (vi) credit and annuity markets are perfect.

What we observe from Figure 7 is that both the additive and multiplicative models (with the same parameter values as before) predict that the historical and projected mortality decline leads to a decrease in the rate of interest. In the additive model the effect is driven by the fact that people live longer and have to save for a longer period of retirement. In the multiplicative model, in addition to this effect, there is the relation between mortality and individual rates of discount that plays a major role. Mortality decline makes people appear less impatient on average (although this is not necessarily the case at all ages), and aggregate savings supply increases. The equilibrium rate of interest decreases accordingly. As a consequence, the multiplicative model predicts a decrease in the interest rate of 3.5 percentage points between 1900 and 2000, while the additive one only predicts a decrease of 0.8 percentage point. Again, we find that opting for the multiplicative model instead of the additive one, leads to radically different predictions regarding the macro economic impact of mortality decline. Historical data provided by Siegel (1992) show risk free real rates of return that are lower than those reported in Figure 7²². However, Siegel's results indicate a long term decrease in the risk free real rate of return that is comparable in magnitude to the one predicted by the multiplicative model²³.

²¹We could have introduced an exogenous technological progress (that is a constant A that depends on the year we consider). But, as we only do comparative statics, that would not change the results.

²²Our simulations rely on the assumptions that annuity markets are perfect and that there are no bequest motives. Both assumptions certainly lead to underestimating the saving supply (and consequently to overestimating the equilibrium rate of interest).

²³An OLS regression on Siegel's results indicates that the risk free real rate of return roughly

Another dimension in which both models lead to radically different prediction is the variation in the value of statistical life along the life cycle. In Figure 8 we plot the value of a statistical life (see Definition 7) for individuals who expect to have a constant consumption profile^{24,25}. Both models give similar (survival weighted) average measures of the value of a statistical life, but this is because they are calibrated as such (see footnote 17). The interesting point is that the two curves shown in Figure 8 have very different shapes. According to the additive model, the value of statistical life declines only very slowly with age. The multiplicative model predicts a much faster decline. That is because ICA makes people less likely to take the risk of losing many years of life. So far, there are very few empirical results on the relation between age and Value of a Statistical Life. The most robust estimates are probably those of Aldy and Viscusi (2004). Their empirical findings cannot be directly compared with Figure 8 since they were computed on a sample of individuals who do not have constant consumption paths. However, Bommier and Villeneuve (2004) show that, when accounting for life cycle variations of consumption, Aldy and Viscusi's empirical estimates are much better approximated by the multiplicative model than the additive one (unless one considers an implausible negative rate of time preference of -0.08 per year). As we can guess from Figure 8, and as it is discussed in greater length by Bommier and Villeneuve, policy recommendations very much depend on which model is used. Compared to the additive model, the multiplicative model values much more the reductions of mortality at young ages.

lost 3.5 percentage points per century in a period that goes from 1800 to 1990.

²⁴The results would be different if we had considered non-constant consumption profiles (as those shown in Figure 3). However, we preferred to make the comparison with constant consumption profiles to make it clear that the differences in the predictions of the two models do not exclusively result from differences in the optimal consumption profile.

²⁵We assume that $c(t) = 1$ for all t . Figure 8, which reports the result in million dollars, assumes that one unit of consumption corresponds to 20000 US\$/year.

9 Conclusion

A general representation of stationary preferences was provided. Some particular specifications were also underlined as corresponding to additional assumptions on individual preferences (Figure 1). This naturally suggests several candidates for lifetime utility function. The simplest choice, and also the most restrictive, involves assuming that individuals have neither pure time preferences nor ICA. Two natural and symmetrical extensions seem then possible: one introducing pure time preferences (which gives the additive model) and another one introducing ICA (as in the multiplicative model). Pure time preferences and ICA may eventually be combined, as in the general recursive form.

The literature on intertemporal choice strongly contrasts with the symmetric picture we drew. While the additive model has been extensively considered, and used as a basis for extensions that do not lie within the general framework we consider²⁶, there is not even a single paper that studies the implication of the multiplicative model when considering uncertain lifetime. Simple simulations that draw on realistic mortality data show however that the multiplicative model may be a very serious alternative to the additive one. Although it rules existence of time preferences, it may end up predicting rates of discount that are of a reasonable magnitude. Moreover it may help to explain some empirical puzzles. With the multiplicative model, the decline of consumption at old ages or the low rate of stockholding of households (and especially of younger households) no longer appear as mysteries or evidence of market imperfections.

There surely are plenty of historical and technical reasons that explain why life cycle theory did not explore the case of multiplicative preferences. In particular, the multiplicative model is an attractive option only when both the finiteness and the randomness of the length of life are accounted for. Therefore it could not emerge from studies that considered that the length of life was infinite or known in advance. Moreover, although the multiplicative model is as simple as

²⁶It is for example the case for the common models with hyperbolic discounting or with habit formation.

the additive one, in terms of degrees of freedom, it is mathematically speaking much less tractable. For example, while it is extremely simple to analytically derive the shape of the optimal consumption in the additive case (with perfect markets), it is impossible to find an explicit solution in the multiplicative case. The solution is closely related to mortality rates and can only be numerically estimated. It is very likely that many economists were not desirous of dealing with technical difficulties of this kind, and just found the additive model more convenient.

Opting for a mathematically convenient theory has an indisputable advantage: it makes economists' lives easier. But the cost may be extremely high. There are many aspects of human behavior that could not be explained with the additive model, while they seem rather natural when accounting for ICA. We mentioned above two facts that were identified as "empirical puzzles": the decline of consumption and the increase in portfolio riskiness along the life cycle. But there are other issues that seem much more important, even though they were not so clearly pointed out as being concomitant with a failure of the current theory. We end the paper by reviewing three of them.

The first one concerns the heterogeneity in discount rates. Women, rich people and non-whites (in the USA) are usually found to be less impatient than men, poor people and white. With the additive model, such heterogeneity can only result from heterogeneity in pure time preferences. Thus, the dominant interpretation is that heterogeneity in impatience reflects fundamental differences in taste, whose origins lie deep in human nature or cultural constructs. Men and women, rich and poor, whites and non-whites would simply have different rationalities. It is not so far a leap from there to stating that individuals from specific groups are more rational than others²⁷. Conversely, a life cycle theory that accounts for ICA would explain a great part of the heterogeneity in discount rates by the heterogeneity in mortality. Women and men, rich and poor, etc. would be different not because of their rationality, but simply because of their

²⁷See the interesting discussion in Peart (2000) about the notion of irrationality in the works of Jevons, Fisher, Marshall and Pigou.

mortality. As argued in Peart and Levy (2003) these diverging interpretations may be used to support fundamentally different ideologies²⁸.

The second issue concerns the effect of mortality changes. The huge decline in mortality rates observed along the last two centuries, as well as the dramatic increase of mortality observed in the regions severely touched by the AIDS epidemic, are about the most significant events in recent human history. Naturally, several papers studied the economic impact of mortality changes²⁹. But practically all of them rely on the additive model, although there is no real empirical support for this model³⁰. Accounting for ICA would radically modify our view of the effect of mortality changes and indirectly our understanding of economic development.

The last issue is about the amount of resources that should be dedicated to increase longevity. This is a central question in our society where medical expenses rise very rapidly. Again, as discussed in Bommier and Villeneuve (2004), the dominant approach in the value of life literature consists in using the additive model, while accounting for ICA would significantly improve the capacity of the theory to fit empirical data and suggest very different policy guidance.

ICA appears to be central for discussing several major social issues simply because it is a key element for modelling rational human behavior. The reason is that human beings are first and foremost mortal. Death being irreversible, the risk of death is akin to a sequence of correlated risks on future individuals' states. The rational response to the risk of death (or, could we say, to the very nature of human existence) has then to crucially depend on ICA.

²⁸Levin (1997) is an extreme example where the (alleged) heterogeneity in time preferences across race is used to support a deeply racist theory.

²⁹See for example Bloom, Canning and Graham (2003), Boucekkine, de la Croix and Licandro (2003), Cavalcanti Ferreira and Pessôa (2003) and Becker, Philipson and Soares (2005).

³⁰To our knowledge, there is no paper that tested the assumption of additive separability of preferences using heterogeneity in mortality. A few tests were implemented using data on consumption smoothing, providing contrasted results. But, in any case, such kinds of test cannot tell whether the additive model is appropriate to study the effects of mortality changes. As discussed further in Bommier (2005), additively separable models might be relatively good to model consumption smoothing and, at the same time, very bad to predict the effect of mortality changes. This would be the case if the product of ICA by lifetime consumption is small but not the product of ICA by the value of a statistical life.

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A Proof of Theorem 1

A.1 Necessary conditions

In preamble of the proof, we show that Axiom 3 implies a weaker continuity condition (Axiom 3' , below) which is commonly used to show that preferences over lotteries (or simple measures) can be represented by a von Neumann-Morgenstern utility function.

Axiom 3' (*Archimedean axiom*) For any $l_1, l_2, l_3 \in L(Z)$ such that $l_1 \succ l_2 \succ l_3$ there exist λ_1 and $\lambda_2 \in (0, 1)$ such that

$$\begin{aligned} \lambda_1 l_1 + (1 - \lambda_1) l_3 &\succ l_2 \\ l_2 &\succ \lambda_2 l_1 + (1 - \lambda_2) l_3 \end{aligned} \tag{16}$$

It is fairly simple to show that:

Lemma 1 *Axiom 3 implies Axiom 3'.*

Proof. Consider $l_1 \succ l_2 \succ l_3$ and define the function h by

$$h \begin{cases} [0, 1] \rightarrow L(Z) \\ \lambda \rightarrow h(\lambda) = \lambda l_1 + (1 - \lambda) l_3 \end{cases}$$

The function h is continuous for the weak topology of measures. Axiom 3 and the continuity of h imply that the sets

$$O_1 = h^{-1}(\{l \in L(Z) | l \succ l_2\}) \text{ and } O_2 = h^{-1}(\{l \in L(Z) | l_2 \succ l\})$$

are open subsets of $[0, 1]$. But $1 \in O_1$ and $0 \in O_2$. Thus $O_1 \cap (0, 1)$ and $O_2 \cap (0, 1)$ are not empty. (16) is satisfied for any $\lambda_1 \in O_1 \cap (0, 1)$ and $\lambda_2 \in O_2 \cap (0, 1)$. ■

Lemma 1 makes it clear than our Axioms 1, 2 and 3 imply that the assumptions of Theorem 8.2 of Fishburn (1970) are fulfilled. Thus, preferences we consider can be represented by a von Neumann-Morgenstern utility function. We only need to show that the von Neumann-Morgenstern utility function has the form given in Theorem 1. The proof is structured as follows: we first deal with the discrete time case and then use Axioms 3 and 4 to extend the result to the continuous time case.

In order to proceed in such a way, we need to define tools that allow us to navigate between the continuous and the discrete time frameworks.

Let us first define the following set:

$$Z_d = \{(x_1, x_2, x_3, \dots) | x_i \in X \text{ and such that}$$

the set of i such that $x_i \neq d$ is empty or of the form $\{1, 2, 3, \dots, p\}$ for some finite $p\}$

The set Z_d is the discrete time analogous of the set Z defined in Section 2.

For any positive integer n we define the function f_n by

$$f_n : \begin{cases} Z_d \rightarrow Z \\ (x_1, x_2, \dots) \rightarrow f_n(x_1, x_2, \dots) \end{cases}$$

where $f_n(x_1, x_2, \dots)$ is defined by:

$$f_n(x_1, x_2, \dots)(t) = x_i \text{ for all } t \in [\frac{1}{2^n}(i-1), \frac{1}{2^n}i[$$

Thus the life $f_n(x_1, x_2, \dots)$ is the life where the individual is in state x_1 during the $\frac{1}{2^n}$ first years of his/her life, in state x_2 during the $\frac{1}{2^n}$ following years, etc.

For any $z \in Z$ and any integer n let us define

$$g_n \begin{cases} Z \rightarrow Z_d \\ z \rightarrow g_n(z) = (z(0), z(\frac{1}{2^n}), \dots, z(\frac{i-1}{2^n}), \dots) \end{cases}$$

Thus $g_n(z)$ is the series that gives the value of z at the beginning of each period of time of width $\frac{1}{2^n}$.

By definition, for any $z_d \in Z_d$ we have $g_n(f_n(z_d)) = z_d$. Note also that for any $z_d = (x_1, x_2, \dots) \in Z_d$ and any function w from X into \mathbb{R} , such that $w(d) = 0$, we have

$$\int_0^{+\infty} f_n(w(z_d))(t)dt = \sum_{i=1}^{+\infty} \frac{1}{2^n} w(z_i) \quad (17)$$

The functions f_n and g_n allow to navigate between the continuous time framework and the discrete time framework with time periods of length $\frac{1}{2^n}$. Intuitively, increasing the integer n by one unit is equivalent to splitting the length of the time periods by two in the discrete time model. A more formal way to express this statement is to note, that by definition of the functions f_n , for any

$(x_1, x_2, x_3, \dots) \in Z_d$ we have:

$$f_{n+1}(x_1, x_1, x_2, x_2, x_3, x_3, \dots) = f_n(x_1, x_2, x_3, \dots)$$

For any integer n we define a preference relation \succeq_n on $L(Z_d)$ by

$$\sum_i \alpha_i \delta_{z_i} \succeq_n \sum_i \beta_i \delta_{y_i} \Leftrightarrow \sum_i \alpha_i \delta_{f_n(z_i)} \succeq \sum_i \beta_i \delta_{f_n(y_i)}$$

The strict preference relation, \succ_n , and the indifference relation \sim_n are defined in the same way.

Lemma 2 *For any n , there exist two functions u_n and v_n from X into \mathbb{R} with $u_n(d) = v_n(d) = 0$ and*

$$u_n(1) \sum_{j=0}^{2^n-1} \exp(-jv_n(1)) = \pm 1 \quad (18)$$

such that the relation of preferences \succeq_n can be represented by the von Neumann-Morgenstern utility function

$$U_n(x_1, x_2, x_3, \dots) = u_n(x_1) + \sum_{j=2}^{+\infty} u_n(x_j) \exp\left(-\sum_{k=1}^{j-1} v_n(x_k)\right) \quad (19)$$

Proof. It is clear that we can use Axioms 1, 2 and 3' to derive similar properties for the preferences \succeq_n . Thus, from Theorem 8.2 of Fishburn (1970), we know that the preferences \succeq_n can be represented by a von Neumann-Morgenstern utility function that we denote U_n . As any positive affine transformation of U_n represents the same preferences as U_n , it is possible to assume without generality loss, that:

$$U_n(d, d, d, \dots) = 0 \quad (20)$$

$$U_n(\underbrace{1, 1, \dots, 1}_{2^n \text{ periods}}, d, d, d, d, \dots) = \pm 1 \quad (21)$$

For any $z = (x_1, x_2, \dots) \in Z_d$ and any $x \in \mathbb{R}^+$ let us define $x * z = (x, x_1, x_2, \dots) \in Z_d$. The stationarity axiom writes:

$$\sum_i \alpha_i \delta_{z_i} \succsim_n \sum_i \beta_i \delta_{y_i} \Leftrightarrow \sum_i \alpha_i \delta_{x * z_i} \succsim_n \sum_i \beta_i \delta_{x * y_i} \quad (22)$$

Since two von Neumann-Morgenstern utility functions represent the same preferences if and only if they are related by a positive affine transformation, (22) implies that for any x there exists two real numbers, $u_n(x)$ and $w_n(x) > 0$, such that

$$U_n(x, x_1, x_2, \dots) = u_n(x) + w_n(x) U_n(x_1, x_2, \dots)$$

Define $v_n = -\log(w_n(x))$ to get:

$$U_n(x, x_1, x_2, \dots) = u_n(x) + \exp(-v_n(x)) U_n(x_1, x_2, \dots) \quad (23)$$

Applying this formula to $(x, x_1, x_2, \dots) = (d, d, d, \dots)$, and using (20), it follows that:

$$u_n(d) = 0 \quad (24)$$

Now, for any $(x_1, x_2, \dots, x_p, d, d, d, \dots) \in Z_d$ let us use (23) and iterate p times. We obtain

$$\begin{aligned} U(x_1, x_2, \dots, x_p, d, d, d, \dots) &= u_n(x_1) + \sum_{j=2}^p u_n(x_j) \exp\left(-\sum_{k=1}^{j-1} v_n(x_k)\right) \\ &\quad + \exp\left(-\sum_{k=1}^p v_n(x_k)\right) U_n(d, d, d, \dots) \end{aligned}$$

which, with (20) and (24), gives (19). Equation (18) follows from (21) and (19). \blacksquare

Now, it remains to extend the result of Lemma 2 to the continuous time case. Let us first relate the functions u_n and v_n that represent the preferences in the discrete time model with period of size $\frac{1}{2^n}$, with the functions u_{n+1} and v_{n+1} that represent the preferences in the discrete time model with periods of size $\frac{1}{2^{n+1}}$. Two

utility functions that represent the same preferences are necessarily identical, up to a positive affine transformation. Thus there exist two scalars A_n and $B_n > 0$, such that for all $(x_1, x_2, \dots) \in Z_d$:

$$U_{n+1}(x_1, x_1, x_2, x_2, \dots) = A_n + B_n U_n(x_1, x_2, \dots) \quad (25)$$

Let us apply this equality to $(x_1, x_1, x_2, x_2, \dots) = (d, d, d, d, \dots)$ to $(x_1, x_1, x_2, x_2, \dots) = (x, x, d, d, d, \dots)$ and to $(x_1, x_1, x_2, x_2, \dots) = (x, x, x, x, d, d, d, \dots)$.

We obtain the following three equations:

$$0 = A_n \quad (26)$$

$$u_{n+1}(x)(1 + e^{-v_{n+1}(x)}) = A_n + B_n u_n(x) \quad (27)$$

$$u_{n+1}(x)(1 + e^{-v_{n+1}(x)})(1 + e^{-2v_{n+1}(x)}) = A_n + B_n u_n(x)(1 + e^{-v_n(x)}) \quad (28)$$

It follows that for all x :

$$\begin{aligned} v_{n+1}(x) &= \frac{1}{2} v_n(x) \\ u_{n+1}(x) &= B_n \frac{u_n(x)}{1 + \exp(-\frac{1}{2} v_n(x))} \end{aligned}$$

Let us now write:

$$\begin{aligned} u_n(1) \sum_{j=0}^{2^n-1} \exp(-j v_n(1)) &= u_n(1) \frac{1 - \exp(-2^n v_n(1))}{1 - \exp(-v_n(1))} \\ &= \frac{u_n(1)}{1 + \exp(-\frac{1}{2} v_n(1))} \frac{1 - \exp(-2^n v_n(1))}{1 - \exp(-\frac{1}{2} v_n(1))} \\ &= \frac{1}{B_n} u_{n+1}(1) \frac{1 - \exp(-2^{n+1} v_{n+1}(1))}{1 - \exp(-v_{n+1}(1))} \\ &= \frac{1}{B_n} u_{n+1}(1) \sum_{j=0}^{2^n} \exp(-j v_{n+1}(1)) \end{aligned}$$

The normalization condition (18) leads to

$$B_n = 1 \quad (29)$$

As a consequence, for all x :

$$\begin{aligned} v_n(x) &= \frac{1}{2^n} v_0(x) \\ u_{n+1}(x) &= \frac{u_n(x)}{1 + \exp(-\frac{1}{2^{n+1}} v_0(x))} \end{aligned}$$

Defining $\hat{u}_n(x) = 2^n u_n(x)$ we have:

$$\hat{u}_{n+1}(x) = \hat{u}_n(x) \frac{2}{1 + \exp(-\frac{1}{2^{n+1}} v_0(x))}$$

and, when $n \rightarrow +\infty$, the sequence $\hat{u}_n(x)$ converges to some limit that we denote $u(x)$. It is clear from (24) that $u(d) = 0$.

Therefore, there exists a function u , with $u(d) = 0$ and a function v such that for all $x \in X$ we have $2^n u_n(x) \rightarrow u(x)$ and $2^n v_n(x) \rightarrow v(x)$. For any $z \in Z$, we define

$$U(z) = \int_0^{+\infty} u(c(t)) \exp\left(-\int_0^t v(c(\tau)) d\tau\right) dt$$

From equation (17), for any $z \in Z_c$, we have

$$U(z) = \lim_{n \rightarrow +\infty} U_n(g_n(z)) \quad (30)$$

We now show that $U(z)$ is a von Neumann-Morgenstern utility function that represents the relation of preferences on $L(Z_c)$. What we need to prove is that for any elements $\sum_i \alpha_i \delta_{z_i}$ and $\sum_i \beta_i \delta_{y_i}$ of $L(Z_c)$ we have:

$$\sum_i \alpha_i \delta_{z_i} \succ \sum_i \beta_i \delta_{y_i} \Leftrightarrow \sum_i \alpha_i U(z_i) > \sum_i \beta_i U(y_i)$$

Let us first prove that:

$$\sum_i \alpha_i \delta_{z_i} \succ \sum_i \beta_i \delta_{y_i} \implies \sum_i \alpha_i U(z_i) > \sum_i \beta_i U(y_i)$$

For any $\lambda \in (0, 1)$ denote $l_\lambda = \lambda \sum_i \alpha_i \delta_{z_i} + (1 - \lambda) \sum_i \alpha_i \delta_{y_i}$. By Axiom 2, $\sum_i \alpha_i \delta_{z_i} \succ l_{\frac{3}{4}} \succ l_{\frac{1}{2}} \succ l_{\frac{1}{4}} > \sum_i \beta_i \delta_{y_i}$

We know that for all i :

$$f_n(g_n(z_i)) \rightarrow z_i \text{ and } f_n(g_n(y_i)) \rightarrow y_i$$

which implies that

$$\sum_i \alpha_i \delta_{f_n(g_n(z_i))} \rightarrow \sum_i \alpha_i \delta_{z_i} \text{ and } \sum_i \beta_i \delta_{f_n(g_n(y_i))} \rightarrow \sum_i \beta_i \delta_{y_i}$$

Thus, from Axiom 3, there exists n_0 such that for all $n \geq n_0$:

$$\sum_i \alpha_i \delta_{f_n(g_n(z_i))} \succ l_{\frac{3}{4}} \succ l_{\frac{1}{2}} \succ l_{\frac{1}{4}} \succ \sum_i \beta_i \delta_{f_n(g_n(y_i))} \quad (31)$$

By Axiom 3' , there exists $\lambda_1, \lambda_2 \in (0, 1)$ such that

$$l_{\frac{3}{4}} \succ (1 - \lambda_1) \sum_i \alpha_i \delta_{f_{n_0}(g_{n_0}(z_i))} + \lambda_1 \sum_i \beta_i \delta_{f_{n_0}(g_{n_0}(y_i))} \succ l_{\frac{1}{2}} \quad (32)$$

$$l_{\frac{1}{2}} \succ (1 - \lambda_2) \sum_i \alpha_i \delta_{f_{n_0}(g_{n_0}(z_i))} + \lambda_2 \sum_i \beta_i \delta_{f_{n_0}(g_{n_0}(y_i))} \succ l_{\frac{1}{4}} \quad (33)$$

For simplicity note:

$$\begin{aligned} \sum_j \gamma_j \delta_{f_{n_0}(g_{n_0}(a_j))} &\equiv (1 - \lambda_1) \sum_i \alpha_i \delta_{f_{n_0}(g_{n_0}(z_i))} + \lambda_1 \sum_i \beta_i \delta_{f_{n_0}(g_{n_0}(y_i))} \\ \sum_j \kappa_j \delta_{f_{n_0}(g_{n_0}(b_j))} &\equiv (1 - \lambda_2) \sum_i \alpha_i \delta_{f_{n_0}(g_{n_0}(z_i))} + \lambda_2 \sum_i \beta_i \delta_{f_{n_0}(g_{n_0}(y_i))} \end{aligned}$$

Equations (31), (32) and (33) imply that for $n \geq n_0$

$$\sum_i \alpha_i \delta_{f_n(g_n(z_i))} \succ \sum_j \gamma_j \delta_{f_{n_0}(g_{n_0}(a_j))} \succ \sum_j \kappa_j \delta_{f_{n_0}(g_{n_0}(b_j))} \succ \sum_i \beta_i \delta_{f_n(g_n(y_i))} \quad (34)$$

Note also that for any $n \geq n_0$, and $x \in Z$, we have $f_n(g_n(f_{n_0}(g_{n_0}(x)))) = f_{n_0}(g_{n_0}(x))$. Therefore, from (34):

$$\sum_i \alpha_i \delta_{g_n(z_i)} \succ_n \sum_j \gamma_j \delta_{g_n(f_{n_0}(g_{n_0}(a_j)))} \succ_n \sum_j \kappa_j \delta_{g_n(f_{n_0}(g_{n_0}(b_j)))} \succ_n \sum_i \beta_i \delta_{g_n(y_i)} \quad (35)$$

But for all x in Z , and all $n > n_0$ we know from (25), (26) and (29) that

$$U_n(g_n(f_{n_0}(g_{n_0}(x)))) = U_{n_0}(g_{n_0}(x))$$

and $U_n(g_n(f_{n_0}(g_{n_0}(x))))$ is therefore independent of n . Thus (35) implies that there exists $\varepsilon > 0$ such that

$$\sum_i \alpha_i U_n(g_n(z_i)) - \sum_i \beta_i U_n(g_n(y_i)) > \varepsilon \text{ for all } n \geq n_0$$

which implies that $\sum_i \alpha_i U(z_i) - \sum_i \beta_i U(y_i) \geq \varepsilon$ and therefore $\sum_i \alpha_i U(z_i) > \sum_i \beta_i U(y_i)$.

Now it only remains to prove that:

$$\sum_i \alpha_i U(z_i) > \sum_i \beta_i U(y_i) \implies \sum_i \alpha_i \delta_{z_i} \succ \sum_i \beta_i \delta_{y_i}$$

From Axiom 4, we know that there exists $\sum_i \alpha_i \delta_{w_i} \succ \sum_i \alpha_i \delta_{z_i}$. Also

$$\sum_i \alpha_i U(z_i) > \sum_i \beta_i U(y_i) \implies \sum_i \alpha_i U(z_i) > (1 - \lambda) \sum_i \beta_i U(y_i) + \lambda \sum_i \alpha_i U(w_i)$$

for some $\lambda \in (0, 1)$. From (30), we know that for some n_0 , we have:

$$\begin{aligned} n > n_0 &\implies \sum_i \alpha_i U_n(g_n(z_i)) > (1 - \lambda) \sum_i \beta_i U_n(g_n(y_i)) + \lambda \sum_i \alpha_i U_n(g_n(w_i)) \\ &\implies \sum_i \alpha_i \delta_{g_n(z_i)} \succ_n (1 - \lambda) \sum_i \beta_i \delta_{g_n(y_i)} + \lambda \sum_i \alpha_i \delta_{g_n(w_i)} \\ &\implies \sum_i \alpha_i \delta_{f_n(g_n(z_i))} \succ (1 - \lambda) \sum_i \beta_i \delta_{f_n(g_n(y_i))} + \lambda \sum_i \alpha_i \delta_{f_n(g_n(w_i))} \end{aligned}$$

And thus by Axiom 3, $\sum_i \alpha_i \delta_{z_i} \succeq (1 - \lambda) \sum_i \beta_i \delta_{y_i} + \lambda \sum_i \alpha_i \delta_{w_i}$. With Axiom 2 that implies that $\sum_i \alpha_i \delta_{z_i} \succ \sum_i \beta_i \delta_{y_i}$ and $U(z)$ is a von Neumann-Morgenstern utility function that represents the relation of preferences on $L(Z_c)$.

A.2 Sufficient conditions

The representation of preferences by a continuous von Neumann-Morgenstern utility function implies that Axioms 1, 2 and 3 are fulfilled. It only remains to show that Axioms 4 and 5 are also fulfilled.

Let us begin with Axiom 4 (the non-satiation assumption). Consider $T > 0$ and $z \in Z_c \cap Z_T$. For any (small) $\varepsilon > 0$ it is possible to construct $z_\varepsilon \in Z_c \cap Z_T$ such that $z_\varepsilon(t) = z(t)$ for all $t \in [0, T - 2\varepsilon]$, $z(t) \leq z_\varepsilon(t) \leq z(t) + 1$ for all $t \in [T - 2\varepsilon, T - \varepsilon]$ and $z_\varepsilon(t) = z(t) + 1$ for all $t \in [T - \varepsilon, T]$. Since $z \in Z_T$, that u and v are continuous and u is increasing, there exist two positive constants K_1 and K_2 such that $u(z_\varepsilon(t)) - u(z(t)) > K_1$ for all $t \in [T - \varepsilon, T]$ and $|v(z(t)) - v(z_\varepsilon(t))| < K_2$ for all $t \in [0, T]$. This implies that:

$$\begin{aligned} U(z_\varepsilon) - U(z) &> e^{(-\int_0^{T-2\varepsilon} v(z(\tau)) d\tau)} \int_{T-2\varepsilon}^T u(z(t)) (1 - e^{2K_2\varepsilon}) e^{(-\int_{T-2\varepsilon}^T v(z_\varepsilon(\tau)) d\tau)} dt \\ &\quad + e^{(-\int_0^{T-2\varepsilon} v(z(\tau)) d\tau)} \int_{T-\varepsilon}^T K_2 e^{(-\int_{T-2\varepsilon}^T v(z_\varepsilon(\tau)) d\tau)} dt \end{aligned}$$

which is positive for ε small enough (the first term may be negative, but is of order ε^2 , while the second term is positive and of order ε). Thus, $\delta_{z_\varepsilon} \succ \delta_z$, for ε small enough, and Axiom 4 is fulfilled.

As for Axiom 5 (the stationarity assumption), it simply follows from the fact that for any $t_0 \in \mathbb{R}^+$ any $c_0 \in F([0, t_0], \mathbb{R}^+)$ and any $z \in Z$

$$U(c_0 *_{t_0} z) = A_{c_0, t_0} + B_{c_0, t_0} U(z)$$

with

$$A_{c_0, t_0} = \int_0^{t_0} u(c(t)) \exp\left(-\int_0^t v(c(\tau)) d\tau\right)$$

and

$$B_{c_0, t_0} = \exp \left(- \int_0^{t_0} v(c(\tau)) d\tau \right) > 0$$

so that one goes from $U(z)$ to $U(c_0 *_{t_0} z)$ by a positive affine transformation.

B Proof of Proposition 1

Using (2) and (3), it is straightforward to check that $1 \Rightarrow 2$. Moreover, $1 \Rightarrow 3$ follows from (5). 4 is explicitly weaker than 3 and, therefore, $3 \Rightarrow 4$.

Let us now prove that $2 \Rightarrow 1$. Denote:

$$A = \{c \in \mathbb{R}^+ | v'(c) = 0\} \text{ and } M = \{c \in \mathbb{R}^+ | u'(c)v(c) - u(c)v'(c) = 0\}$$

By definition (and because of Assumption 1) preferences are additive if and only if $A = \mathbb{R}^+$ and multiplicative if and only if $M = \mathbb{R}^+$. The rate of time preference at time t is given by:

$$RTP_t = -\frac{d}{dt} \left(\log \frac{\partial U(c, T)}{\partial c(t)} \right) |_{c'(t)=0} = -\lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} \log \left(\frac{\frac{\partial U(c, T)}{\partial c(t+\varepsilon)}}{\frac{\partial U(c, T)}{\partial c(t)}} \right) \right) |_{c'(t)=0}$$

and therefore:

$$2 \Rightarrow \text{for any } t_1 \neq t \text{ we have } \frac{\partial RTP_t}{\partial c(t_1)} = 0$$

But, by derivation of (5), for $t_1 \in (t, T)$:

$$\begin{aligned} \frac{\partial RTP_t}{\partial c(t_1)} &= v'(c(t)) [v(c(t))u'(c(t)) - v'(c(t))u(c(t))] \exp \left(- \int_t^{t_1} v(c(\tau)) d\tau \right) \\ &\quad \times \frac{u'(c(t_1)) - v'(c(t_1)) \int_{t_1}^T u(c(\tau_1)) e^{-\int_{t_1}^{\tau_1} v(c(\tau)) d\tau} d\tau_1}{\left[u'(c(t)) - v'(c(t)) \int_t^T u(c(\tau_1)) e^{-\int_t^{\tau_1} v(c(\tau)) d\tau} d\tau_1 \right]^2} \end{aligned}$$

Assumption 2 implies that:

$$u'(c(t_1)) - v'(c(t_1)) \int_{t_1}^T u(c(\tau_1)) e^{-\int_{t_1}^{\tau_1} v(c(\tau)) d\tau} d\tau_1 > 0 \text{ for all } (c, T) \text{ and } t < T \quad (36)$$

Thus it is clear that $\frac{\partial RTP_t}{\partial c(t_1)} = 0$ if and only if $v'(c(t)) = 0$ or $u'(c(t))v(c(t)) = u(c(t))v'(c(t))$. Therefore:

$$2 \Rightarrow A \cup M = \mathbb{R}^+$$

We now prove that $A \cup M = \mathbb{R}^+$, together with Assumptions 1 and 2, implies that $A = \emptyset$ (and $M = \mathbb{R}^+$) or $A = \mathbb{R}^+$.

Denote \mathbb{C}_A the complement of A . Because of Assumption 1, \mathbb{C}_A is open. Assume that \mathbb{C}_A is not empty and consider $c^* \in \mathbb{C}_A$. Note I_c the largest open interval that is included in \mathbb{C}_A and contains c^* . Since $A \cup M = \mathbb{R}^+$ we know that $\mathbb{C}_A \subset M$ and $I_c \subset M$. Thus, there must exist a constant k_I such that $v(c) = k_I u(c)$ for all $c \in I_c$. By continuity (Assumption 1), such a relation must extend to $\overline{I_c}$, the closure of I_c . This implies that, $v'(c) = k_I u'(c)$ for all $c \in \overline{I_c}$. The constant k_I is necessarily different from zero (by definition of A) and, with Assumption 2, this implies that $\overline{I_c} \subset \mathbb{C}_A$ (otherwise there would exist $c \in \mathbb{R}^+$ such that $v'(c) = u'(c) = 0$ which would contradict (36)). But as I_c is the largest open interval of \mathbb{C}_A that contains c^* , it is necessarily the case that $\overline{I_c} \subset I_c$. Therefore $I_c = \mathbb{R}^+$ and $\mathbb{C}_A = \mathbb{R}^+$. Thus

$$(A \cup M = \mathbb{R}^+) \Rightarrow 1 \tag{37}$$

It only remains to prove that $4 \Rightarrow 1$. From (5) we know that for all constant consumption paths:

$$RTP_t = \frac{u'(c)v(c) - v'(c)u(c)}{u'(c) + v'(c)u(c) \left(\frac{e^{-Tv(c)} - e^{-tv(c)}}{v(c)} \right)}$$

Thus $\frac{d}{dt} RTP_t = 0$ if and only if $v'(c)u(c) = 0$ or $u'(c)v(c) - v'(c)u(c) = 0$. In other words and $4 \Rightarrow A \cup M \cup \{c | u(c) = 0\} = \mathbb{R}^+$. Note that there exists at most one point c such that $u(c) = 0$ (otherwise Assumption 2 would not be fulfilled). Since $A \cup M$ is closed we have $A \cup M \cup \{c | u(c) = 0\} = \mathbb{R}^+ \Rightarrow A \cup M = \mathbb{R}^+$. Using (37), we find that $4 \Rightarrow 1$.

C Proof of Theorem 2

By derivation of equation (13) we get:

$$\begin{aligned} \frac{\partial E_{\mu,0}U(c,.)}{\partial c(t_1)} &= \exp\left(-\int_0^{t_1} v(c(\tau))d\tau\right) \\ &\times \left[s(t_1)u'(c(t_1)) - v'(c(t_1)) \int_{t_1}^{+\infty} s(\tau_1)u(c(\tau_1)) \exp\left(-\int_{t_1}^{\tau_1} v(c(\tau))d\tau\right) d\tau_1\right] \end{aligned} \quad (38)$$

and for any $t_2 < t_1$ we obtain:

$$\frac{\partial^2 E_{\mu,0}U(c,.)}{\partial c(t_1)\partial c(t_2)} = -v'(c(t_2)) \frac{\partial E_{\mu,0}U(c,.)}{\partial c(t_1)}$$

from which, it follows that $\rho_{\mu,t} = v'(c(t))$ and, consequently, (14).

From (38):

$$\begin{aligned} -\frac{d}{dt}(\log \frac{\partial E_{\mu,0}U(c,.)}{\partial c(t)}) &= \frac{\mu(t)u'(c(t)) + v(c(t))u'(c(t)) - v'(c(t))u(c(t))}{u'(c(t)) - v'(c(t)) \int_t^{+\infty} \frac{s(\tau_1)}{s(t)}u(c(\tau_1)) \exp\left(-\int_t^{\tau_1} v(c(\tau))d\tau\right) d\tau_1} \\ &= \mu(t) + \frac{v(c(t))u'(c(t)) - v'(c(t))u(c(t))}{u'(c(t)) - v'(c(t)) \int_t^{+\infty} \frac{s(\tau_1)}{s(t)}u(c(\tau_1)) \exp\left(-\int_t^{\tau_1} v(c(\tau))d\tau\right) d\tau_1} \\ &\quad + \mu(t) \frac{v'(c(t)) \int_t^{+\infty} \frac{s(\tau_1)}{s(t)}u(c(\tau_1)) \exp\left(-\int_t^{\tau_1} v(c(\tau))d\tau\right) d\tau_1}{u'(c(t)) - v'(c(t)) \int_t^{+\infty} \frac{s(\tau_1)}{s(t)}u(c(\tau_1)) \exp\left(-\int_t^{\tau_1} v(c(\tau))d\tau\right) d\tau_1} \end{aligned}$$

That leads to (15), once we remark that:

$$-\frac{\frac{\partial E_0U(c,.)}{\partial \mu(t)}}{\frac{\partial E_0U(c,.)}{\partial c(t)}} = \frac{\int_t^{+\infty} \frac{s(\tau_1)}{s(t)}u(c(\tau_1)) \exp\left(-\int_t^{\tau_1} v(c(\tau))d\tau\right) d\tau_1}{u'(c(t)) - v'(c(t)) \int_t^{+\infty} \frac{s(\tau_1)}{s(t)}u(c(\tau_1)) \exp\left(-\int_t^{\tau_1} v(c(\tau))d\tau\right) d\tau_1}$$

■

Figure 1: The set of stationary preferences

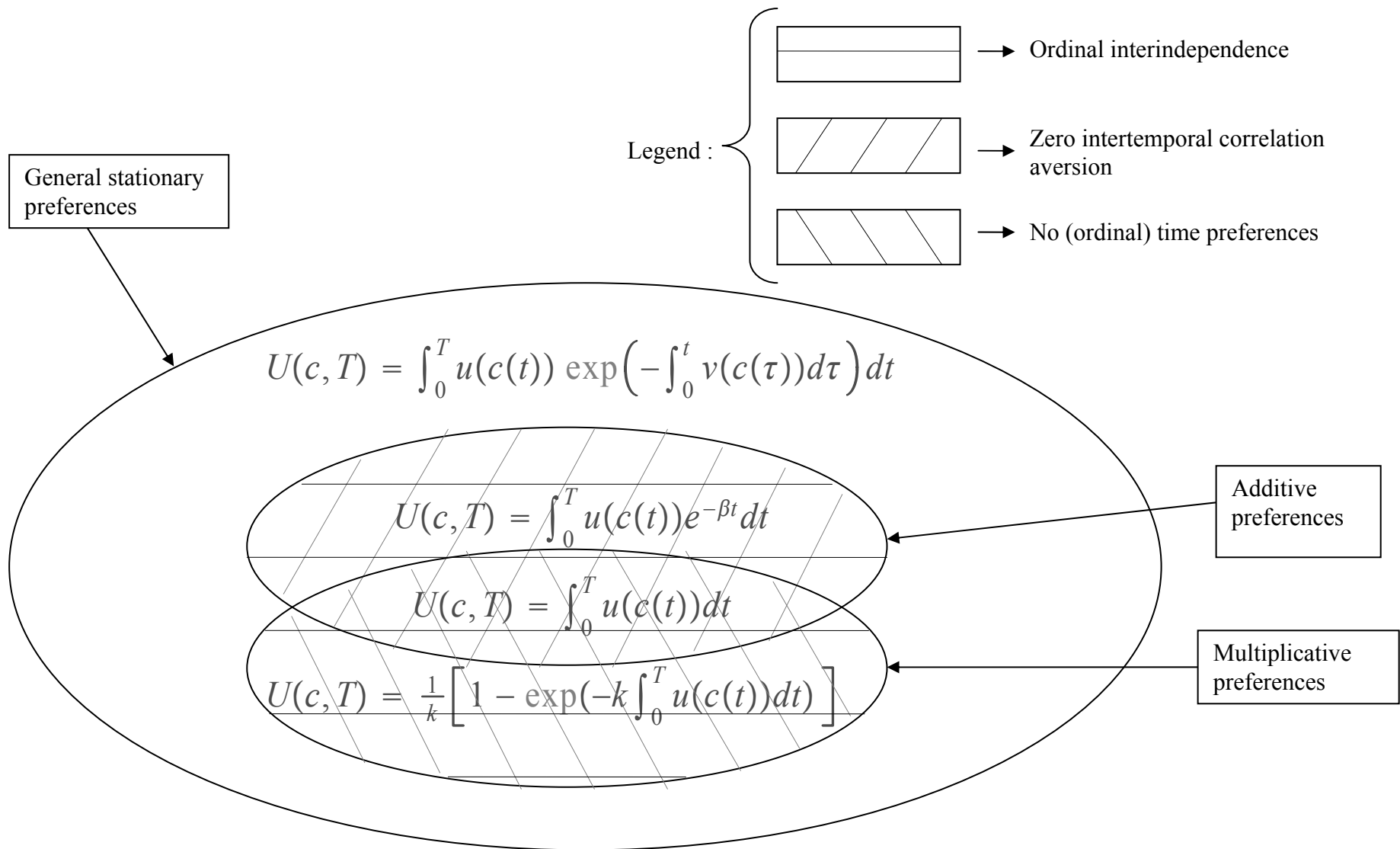
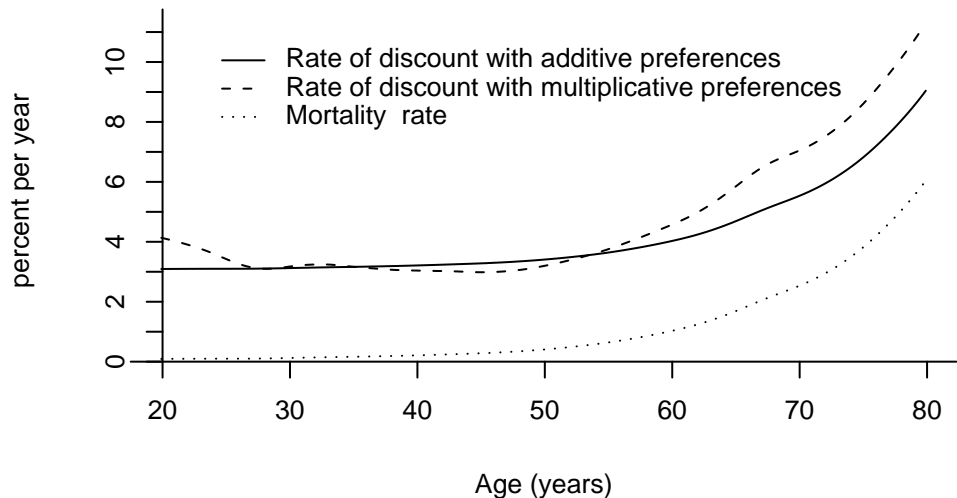
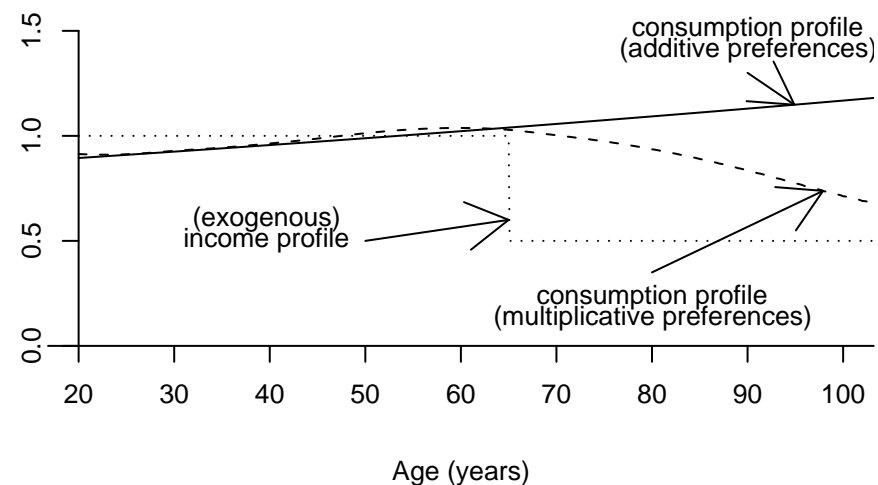


Figure 2: Rate of discount with additive and multiplicative preferences



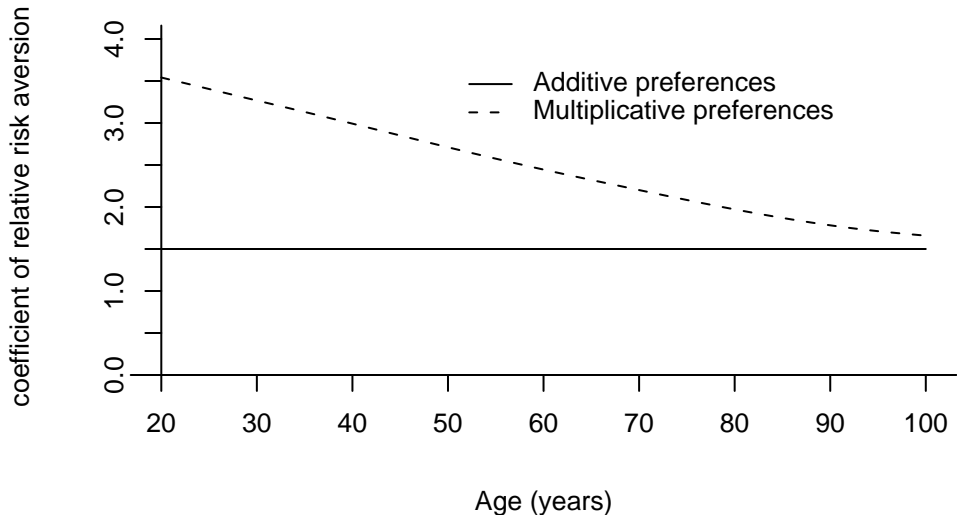
Mortality data from the 2000 US lifetable. Preference parameters as in footnote 17.

Figure 3: Life cycle consumption smoothing with perfect credit and annuity markets



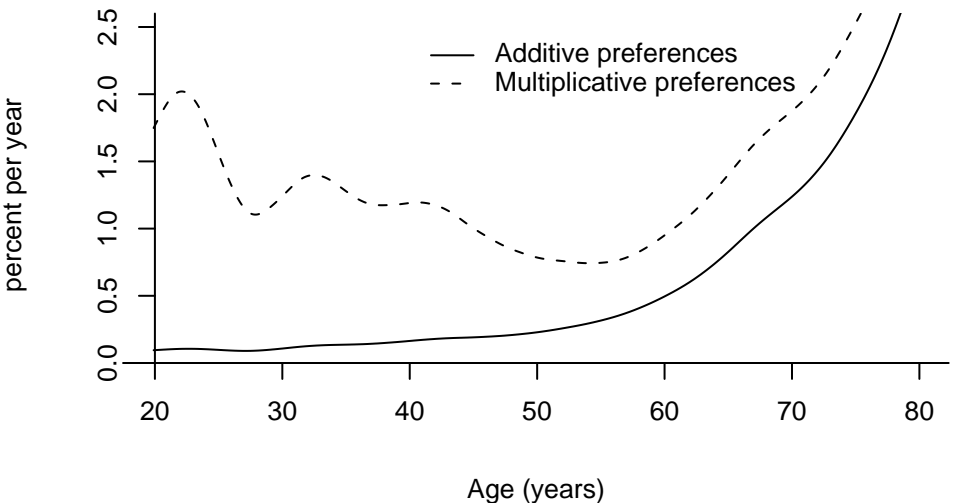
Mortality data from the 2000 US lifetable. Preference parameters as in footnote 17.

Figure 4: Relative risk aversion with respect to remaining lifetime income



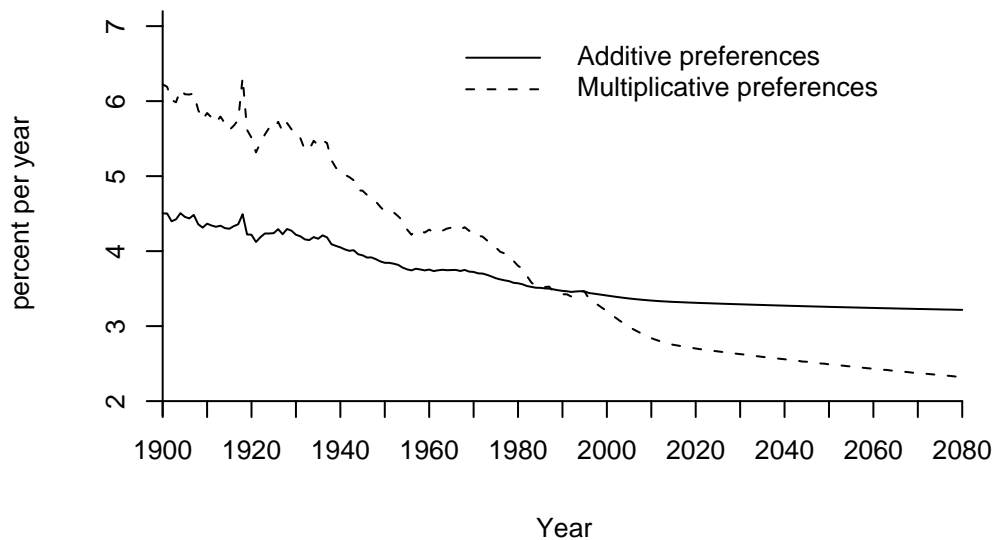
Mortality data from the 2000 US lifetable. Preference parameters as in footnote 17.

Figure 5: Difference between the male and female rates of discount



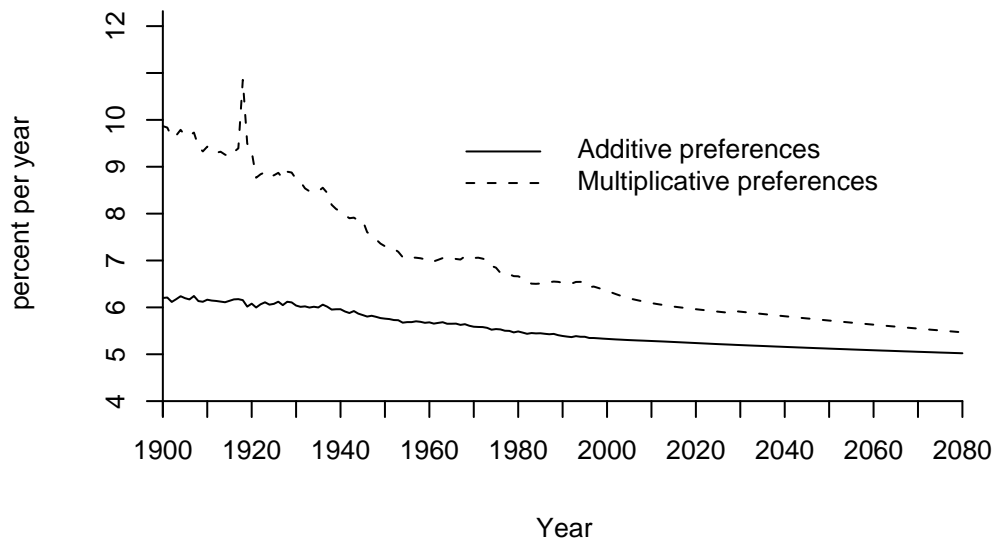
Mortality data from the 2000 gender-specific US lifetables. Preferences parameters as in footnote 17.

Figure 6: Rate of discount of a 50 year old individual



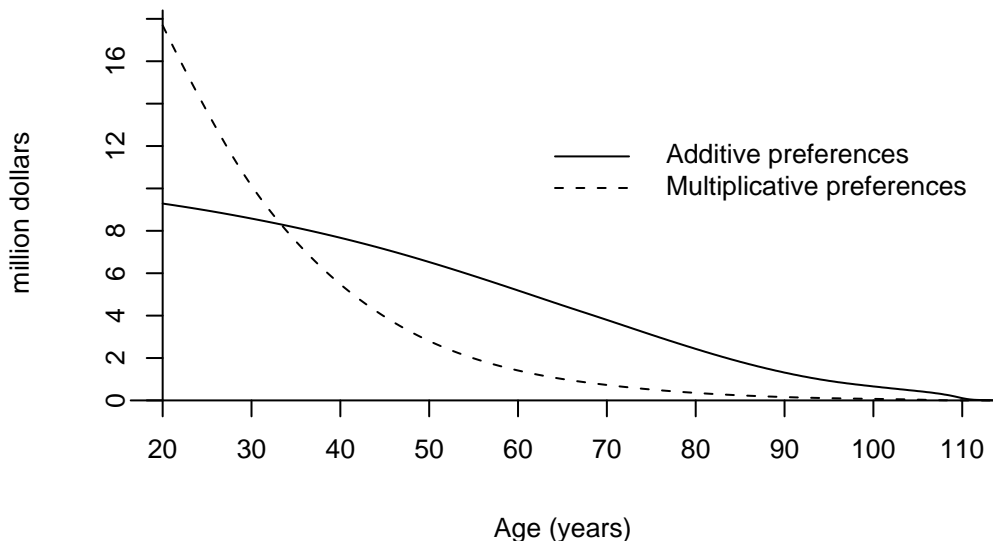
Historical and projected US mortality. Preference parameters as in footnote 17.

Figure 7: Rate of interest in steady-state general equilibria



Historical and projected US mortality. Preference parameters as in footnote 17.

Figure 8: Value of a statistical life



Mortality data from the 2000 US life table. Preference parameters as in footnote 17. Yearly consumption of 20000 dollars